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GENERAL INFORMATION

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Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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# THE OLYMPIAD CORNER

No. 129

R.E. WOODROW

*All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

The first item today is the 32nd I.M.O. held in Sigtuna, Sweden, July 17–18, 1991. My sources this year are Bruce Shawyer, Memorial University of Newfoundland, who was an observer for Canada and did not actually take part in the deliberations; a press release of the contest committee of the MAA; and Andy Liu, The University of Alberta, who did not attend the meeting, but who has excellent contacts. The results from him are relayed from Professor Pak-Hong Cheung, leader of the Hong Kong team. It would have been nice to include some more local colour from official representatives of the Canadian team, but their email messages to me went off into the ether.

This year a record 312 students from 54 countries were officially recorded as participants in the contest. The team from North Korea was disqualified because of overly striking similarities between the official solution of one problem and five of the submitted solutions. There were seven countries that did not send a full team of six persons. They were Denmark (5), Luxembourg (2), Switzerland (1), Trinidad & Tobago (4), Tunisia (4), Cyprus (4) and The Philippines (4).

The six problems of the competition were assigned equal weights of seven points each (the same as in the last 10 I.M.O.'s) for a maximum possible individual score of 42 (and a maximum possible team score of 252). For comparisons see the last 10 I.M.O. reports in [1981: 220], [1982: 223], [1983: 205], [1984: 249], [1985: 202], [1986: 169], [1987: 207], [1988: 193], [1989: 193], and [1990: 193].

This year first place (gold) medals were awarded to the twenty students who scored 39 or higher. Second place (silver) medals were awarded to the 51 students whose scores were in the range 31–38, and third place (bronze) medals were awarded to the 84 students whose scores were in the range 19 to 30. In addition, as in recent years, Honourable Mention was given to any candidate who did not qualify for a medal, but who scored 7 out of 7 on one or more problems. There were nine perfect scores, four of which were by members of the winning Soviet team. One of these was by Evgeniya Malinnikova. This makes her fourth gold medal and her second perfect score at an I.M.O.!

Congratulations to the gold medal winners:

| Student | Country | Score |
|---|---|---|
| Ambajnis, Andris | USSR | 42 |
| Fryers, Michael | UK | 42 |
| Lafforgue, Vincent | France | 42 |
| Lakos, Gyula | Hungary | 42 |
| Malinnikova, Evgeniya | USSR | 42 |
| Moroianu, Sergiu | Romania | 42 |
| Perlin, Alexander | USSR | 42 |

| | | |
|---|---|---|
| Temkin, Mikhail | USSR | 42 |
| Wei, Luo | China | 42 |
| Bănică, Teodor | Romania | 41 |
| Hoffmann, Norbert | Germany | 41 |
| Lizhao, Zhany | China | 41 |
| Újvári-Menyhárt, Zoltan | Hungary | 41 |
| Ionescu, Alexandru Dan | Romania | 40 |
| Noohi, Behrang | Iran | 40 |
| Shaoyu, Wang | China | 40 |
| Song, Wang | China | 40 |
| Goldberg, Ian | Canada | 39 |
| Kassaie, Payman L. | Iran | 39 |
| Rosenberg, Joel | USA | 39 |

Here are the problems from this year's I.M.O. Competition. Solutions to these problems along with those of the 1991 USA Mathematical Olympiad, will appear in a booklet entitled *Mathematical Olympiads 1991* which may be obtained for a small charge from:

Dr. W.E. Mientka
Executive Director
MAA Committee on H.S. Contests
917 Oldfather Hall
University of Nebraska
Lincoln, Nebraska, USA 68588.

## 32nd INTERNATIONAL MATHEMATICAL OLYMPIAD

Sigtuna, Sweden

First Day — July 17, 1991 ($4\frac{1}{2}$ hours)

**1.** Given a triangle $ABC$, let $I$ be the centre of its inscribed circle. The internal bisectors of the angles $A$, $B$, $C$ meet the opposite sides in $A'$, $B'$, $C'$ respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}\,.$$

**2.** Let $n > 6$ be an integer and $a_1, a_2, \ldots, a_k$ be all the natural numbers less than $n$ and relatively prime to $n$. If

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that $n$ must be either a prime number or a power of 2.

**3.** Let $S = \{1, 2, \ldots, 280\}$. Find the smallest integer $n$ such that each $n$-element subset of $S$ contains five numbers which are pairwise relatively prime.

Second Day — July 18, 1991 (4 $\frac{1}{2}$ hours)

**4.** Suppose $G$ is a connected graph with $k$ edges. Prove that it is possible to label the edges $1, 2, 3, \ldots, k$ in such a way that at each vertex which belongs to two or more edges the greatest common divisor of the integers labelling those edges is equal to 1.

[A *graph* $G$ consists of a set of points, called *vertices*, together with a set of *edges* joining certain pairs of distinct vertices. Each pair of vertices $u$, $v$ belongs to at most one edge. The graph $G$ is *connected* if for each pair of distinct vertices $x$, $y$ there is some sequence of vertices $x = v_0, v_1, v_2, \ldots, v_m = y$ such that each pair $v_i, v_{i+1}$ $(0 \leq i < m)$ is joined by an edge of $G$.]

**5.** Let $ABC$ be a triangle and $P$ an interior point in $ABC$. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.

**6.** An infinite sequence $x_0, x_1, x_2, \ldots$ of real numbers is said to be *bounded* if there is a constant $C$ such that $|x_i| \leq C$ for every $i \geq 0$. Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that

$$|x_i - x_j| \cdot |i - j|^a \geq 1$$

for every pair of distinct non-negative integers $i, j$.

*

As the I.M.O. is officially an individual event, the compilation of team scores is unofficial, if inevitable. Team scores are obtained by adding up the individual scores of the members. These totals, as well as a breakdown of the medals awarded by country, is given in the following table.

Congratulations to the USSR for winning this year.

| Rank | Country | Score (Max 252) | Prizes 1st | 2nd | 3rd | Total Prizes |
|---|---|---|---|---|---|---|
| 1 | USSR | 241 | 4 | 2 | - | 6 |
| 2 | China | 231 | 4 | 2 | - | 6 |
| 3 | Romania | 225 | 3 | 2 | 1 | 6 |
| 4 | Germany | 222 | 1 | 5 | - | 6 |
| 5 | USA | 212 | 1 | 4 | 1 | 6 |
| 6 | Hungary | 209 | 2 | 3 | 1 | 6 |
| 7 | Bulgaria | 192 | - | 3 | 3 | 6 |
| 8–9 | Iran | 191 | 2 | 1 | 2 | 5 |
| 8–9 | Vietnam | 191 | - | 4 | 2 | 6 |
| 10 | India | 187 | - | 3 | 3 | 6 |
| 11 | Czechoslovakia | 186 | - | 4 | 1 | 5 |
| 12 | Japan | 180 | - | 3 | 3 | 6 |
| 13 | France | 175 | 1 | 1 | 4 | 6 |
| 14 | Canada | 164 | 1 | 2 | 2 | 5 |
| 15 | Poland | 161 | - | 2 | 4 | 6 |
| 16 | Yugoslavia | 160 | - | 2 | 3 | 5 |
| 17 | South Korea | 151 | - | 1 | 4 | 5 |
| 18–19 | Austria | 142 | - | 2 | 3 | 5 |
| 18–19 | United Kingdom | 142 | 1 | - | 2 | 3 |

| Rank | Country | Score | Prizes | | | Total Prizes |
|---|---|---|---|---|---|---|
| 20 | Australia | 129 | - | - | 3 | 3 |
| 21 | Sweden | 125 | - | 2 | 1 | 3 |
| 22 | Belgium | 121 | - | - | 3 | 3 |
| 23 | Israel | 115 | - | 1 | 2 | 3 |
| 24 | Turkey | 111 | - | - | 4 | 4 |
| 25 | Thailand | 103 | - | 1 | 1 | 2 |
| 26 | Colombia | 96 | - | - | 2 | 2 |
| 27–28 | Argentina | 94 | - | - | 3 | 3 |
| 27–28 | Singapore | 94 | - | 1 | 1 | 2 |
| 29–30 | Hong Kong | 91 | - | - | 2 | 2 |
| 29–30 | New Zealand | 91 | - | - | 2 | 2 |
| 31–32 | Morocco | 85 | - | - | 1 | 1 |
| 31–32 | Norway | 85 | - | - | 3 | 3 |
| 33 | Greece | 81 | - | - | 2 | 2 |
| 34 | Cuba | 80 | - | - | 2 | 2 |
| 35 | Mexico | 76 | - | - | 1 | 1 |
| 36 | Italy | 74 | - | - | 1 | 1 |
| 37–38 | Brazil | 73 | - | - | 1 | 1 |
| 37–38 | The Netherlands | 73 | - | - | 1 | 1 |
| 39 | Tunisia (4) | 69 | - | - | 2 | 2 |
| 40–41 | Finland | 66 | - | - | 1 | 1 |
| 40–41 | Spain | 66 | - | - | 1 | 1 |
| 42 | Philippines (4) | 64 | - | - | 2 | 2 |
| 43 | Denmark (5) | 49 | - | - | - | - |
| 44 | Ireland | 47 | - | - | - | - |
| 45 | Trinidad & Tobago (4) | 46 | - | - | - | - |
| 46 | Portugal | 42 | - | - | - | - |
| 47 | Mongolia | 33 | - | - | - | - |
| 48–49 | Indonesia | 30 | - | - | - | - |
| 48–49 | Luxembourg (2) | 30 | - | - | 1 | 1 |
| 50–51 | Switzerland (1) | 29 | - | - | 1 | 1 |
| 50–51 | Iceland | 29 | - | - | 1 | 1 |
| 52 | Cyprus (4) | 25 | - | - | - | - |
| 53 | Algeria | 20 | - | - | - | - |
| 54 | Macao | 18 | - | - | - | - |

This year the Canadian team slipped back from eleventh to fourteenth place, but put in a good performance. The team members, scores, and the leader of the Canadian team were

| | | |
|---|---|---|
| Ian Goldberg | 39 | Gold |
| J.P. Grossman | 36 | Silver |
| Adam Logan | 32 | Silver |
| Peter Milley | 22 | Bronze |
| Mark von Raamsdonk | 21 | Bronze |
| Ka-Ping Yee | 14 | Honourable Mention. |

Team leader: Professor Georg Gunther, Sir Wilfred Grenfell College.

The USA team slipped from 3rd to 5th. The results for its members were

| | |
|---|---|
| Joel Rosenberg | Gold |
| Kiran Kedlaya | Silver |
| Robert Kleinberg | Silver |
| Lenhard Ng | Silver |
| Michael Sunitsky | Silver |
| Ruby Breydo | Bronze |

The team leaders were Professors C. Rousseau, Memphis State University, and Dan Ullman of George Washington University.

* * *

Next we turn to "Archive" problems from the 1988 numbers of *Crux*.

**3.** [1988: 101] *Tenth Atlantic Provinces Mathematics Competition.*
Suppose $S(X)$ is given by

$$S(X) = X(1 + X^2(1 + X^3(1 + X^4(1 + \dots)).$$

Is $S(1/10)$ rational?

*Solution by Murray S. Klamkin, University of Alberta.*

We show that $S(1/n)$ is irrational for any positive integer $n > 1$ where $S(X)$ is the power series

$$X + X^3 + X^6 + \cdots = \sum X^{n(n+1)/2}.$$

Any rational number when expanded into a "decimal" in any base must eventually be periodic. Here $S(1/n)$ in base $n$ is $.1010010001\dots$ . By considering the blocks of successive zero digits, it is clear the number cannot be periodic.

**8.** [1988: 102] *Tenth Atlantic Provinces Mathematics Competition.*
Find the sum of the infinite series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$$

where the terms are reciprocals of integers divisible only by the primes 2 or 3.

*Correction and generalization by Murray S. Klamkin, University of Alberta.*

One must add "except for the first term 1" at the end of the statement.

If we replace reciprocals of the numbers of the form $2^k3^\ell$ with reciprocals of numbers of the form $P_1^{k_1}P_2^{k_2}\dots P_n^{k_n}$ where $P_1, P_2, \dots, P_n$ are distinct primes, then the sum is given by

$$\left(\sum_{k=0}^{\infty}\frac{1}{P_1^k}\right)\left(\sum_{k=0}^{\infty}\frac{1}{P_2^k}\right)\cdots\left(\sum_{k=0}^{\infty}\frac{1}{P_n^k}\right) = \frac{P_1}{P_1 - 1}\cdot\frac{P_2}{P_2 - 1}\cdot\dots\cdot\frac{P_n}{P_n - 1}\,.$$

For the given problem this gives $\frac{2}{1}\cdot\frac{3}{2} = 3$.

[*Editor's Note:* The problem was also solved by John Morvay, Springfield, Missouri.]

* * *

Next is an interesting solution to one of the problems of the 29th I.M.O. in Australia. Normally we don't publish solutions to these problems as the "official" solutions are readily available. This warrants an exception.

**6.** [1988: 197] *29th I.M.O.*

Let $a$ and $b$ be positive integers such that $ab+1$ divides $a^2+b^2$. Show that

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer.

*Solution by Joseph Zaks, The University of Haifa, Israel.*

If $(a_i)$ is a doubly-infinite sequence (i.e., $-\infty < i < \infty$) of reals, satisfying

(1) $a_{i+2} = na_{i+1} - a_i$ for all $i$, and

(2) $\dfrac{a_i^2 + a_{i+1}^2}{a_i a_{i+1} + 1} = n$ for *one* value of $i$,

then (2) is satisfied for *all* values of $i$.

To see it, observe that

$$\frac{a_{i+2}^2 + a_{i+1}^2}{a_{i+2}a_{i+1}+1} = \frac{(na_{i+1}-a_i)^2 + a_{i+1}^2}{(na_{i+1}-a_i)a_{i+1}+1} = \frac{(a_{i+1}^2 + a_i^2) + n(na_{i+1}^2 - 2a_{i+1}a_i)}{(a_{i+1}a_i + 1) + (na_{i+1}^2 - 2a_{i+1}a_i)} = n$$

where the last equality uses the assumption (2). Observe that $a_{i+2} = na_{i+1} - a_i$ is equivalent to $a_i = na_{i+1} - a_{i+2}$, thus it follows that (2) holds for all $i$.

To solve the problem, suppose $a$ and $b$ are integers, for which $(a^2+b^2)/(ab+1) = n$ is an integer. If $a = b$, then it follows easily that $a = b = 1$, and the assertion is elementary. Otherwise, say $1 < a < b$, and let $a_1 = a$ and $a_2 = b$, and define $(a_i)$ by $a_{i+2} = na_{i+1} - a_i$ and by $a_i = na_{i+1} - a_{i+2}$. It follows easily (since $n > 1$) that the doubly infinite sequence of integers $a_i$ is monotonic increasing (in $i$), thus in going backwards with $i$, $a_i$ tends to $-\infty$. Thus $a_i$ becomes negative for suitable $i$. However, since $n$ is positive, there is no $j$ for which $a_j > 0$ and $a_{j-1} < 0$, because of (2). Thus $a_{j-1} = 0$ for some $j$, implying (use (2) with $i = j-1$) that $n = a_j^2$.

As a by-product, we can get all the triples $(a,b,n) \neq (1,1,1)$ of integers satisfying $(a^2+b^2)/(ab+1) = n$. Such $a$ and $b$ will be any two consecutive terms of the sequence $(a_i)$ satisfying $a_0 = 0$, $a_1 = m$ and defined by (1) where $n = m^2$. The characteristic equation of (1) is $x^2 - m^2x + 1 = 0$, which has the solutions

$$x = \frac{m^2 \pm \sqrt{m^4-4}}{2};$$

thus, for suitable $\alpha$ and $\beta$,

$$a_i = \alpha\left(\frac{m^2+\sqrt{m^4-4}}{2}\right)^i + \beta\left(\frac{m^2-\sqrt{m^4-4}}{2}\right)^i$$

for all $i \geq 0$. From $a_0 = 0$ and $a_1 = m$ we get

$$a_i = \frac{m}{\sqrt{m^4-4}}\left(\frac{m^2+\sqrt{m^4-4}}{2}\right)^i - \frac{m}{\sqrt{m^4-4}}\left(\frac{m^2-\sqrt{m^4-4}}{2}\right)^i$$

and the general solution is $(a_i, a_{i+1}, m^2)$ for $i = 0, 1, \ldots$ .

* * *

Now we turn to problems from the January 1990 number of the Corner. First let me apologize for leaving Murray Klamkin's name off the list [1991: 235] of people who solved problem 5 of the Singapore Mathematical Society Interschool Mathematical Competition. His solution was stuck to one for the Chinese contests discussed below.

**1.** [1990: 5] *First Selection Test of the Chinese I.M.O. Team 1988.*

What necessary and sufficient conditions must real numbers $A, B, C$ satisfy in order that

$$A(x-y)(x-z) + B(y-z)(y-x) + C(z-x)(z-y)$$

is nonnegative for all real numbers $x$, $y$ and $z$?

*Solution by Murray S. Klamkin, University of Alberta.*

Expanding out, we get

$$Ax^2 + By^2 + Cz^2 + yz(A-B-C) + zx(B-C-A) + xy(C-A-B) \geq 0.$$

As is known, the corresponding matrix

$$\begin{bmatrix} A & \alpha & \beta \\ \alpha & B & \gamma \\ \beta & \gamma & C \end{bmatrix}$$

where $2\alpha = C-A-B, 2\beta = B-C-A, 2\gamma = A-B-C$, must be nonnegative definite. The necessary and sufficient conditions for this are that all the principal minors be nonnegative. Thus $A$, $B$, $C$ must be $\geq 0$. All the principal 2nd order minors are the same and equal

$$\frac{1}{4}(2(BC+CA+AB) - A^2 - B^2 - C^2).$$

Since this expression corresponds to 4 times the square of the area of a triangle of sides $\sqrt{A}$, $\sqrt{B}$, $\sqrt{C}$, these three numbers must satisfy the triangle inequalities. The third order minor, which is the determinant of the matrix, is 0 (just add the 2nd and 3rd row to the top one). Summarizing, the necessary and sufficient conditions are that $\sqrt{A}$, $\sqrt{B}$, $\sqrt{C}$ are possible sides of a triangle, possibly degenerate.

*Comment:* On replacing $A, B, C$ by $a^2, b^2, c^2$, respectively, where $a, b, c$ are sides of a triangle, the given form can be rewritten as

$$(ax)^2 + (by)^2 + (cz)^2 - 2(by)(cz)\cos\alpha - 2(cz)(ax)\cos\beta - 2(ax)(by)\cos\gamma$$

where $\alpha, \beta, \gamma$ are the angles of the triangle. Since $x, y, z$ are arbitrary, so are $ax, by, cz$ so these can be replaced by arbitrary $u, v, w$ to give

$$u^2 + v^2 + w^2 - 2vw\cos\alpha - 2wu\cos\beta - 2uv\cos\gamma.$$

For a generalization of this nonnegative form and an expression of it as a sum of two squares, see *Crux* 1201 [1988: 90].

[*Editor's Note:* Seung-Jin Bang, Seoul, Republic of Korea, also sent in a solution that gave $A, B, C \geq 0$ and $A^2 + B^2 + C^2 \leq 2(AB + BC + CA)$ as necessary and sufficient conditions, but that did not extend to the observation about triangles. He sets $X = x - y$ and $Y = x - z$, and the expression becomes

$$AXY + B(X - Y)X + C(-Y)(X - Y) = BX^2 + (A - B - C)XY + CY^2.$$

Setting $X = 0$, $Y = 0$ and then $X = Y$ gives $A, B, C \geq 0$. Viewing the expression as a quadratic in $X$ which is non-negative gives $(A-B-C)^2 - 4BC \leq 0$ so that $A^2+B^2+C^2 \leq 2(AB + BC + CA)$.]

**2.** [1990: 5] *First Selection Test of the Chinese I.M.O. Team 1988.*

Determine all functions $f$ from the rational numbers to the complex numbers such that

(i) $f(x_1 + x_2 + \cdots + x_{1988}) = f(x_1)f(x_2)\ldots f(x_{1988})$

for all rational numbers $x_1, x_2, \ldots, x_{1988}$, and

(ii) $\overline{f(1988)}f(x) = f(1988)\overline{f(x)}$

for all rational numbers $x$, where $\overline{z}$ denotes the complex conjugate of $z$.

*Solution by Seung-Jin Bang, Seoul, Republic of Korea.*

Suppose $f \not\equiv 0$. Since $f(x) = f(x)f(0)^{1987}$, we have $(f(0))^{1987} = 1$. Note that $f(x + y) = f(x)f(y)(f(0))^{1986}$. Let $g(x) = f(x)/f(0)$. Then

$$g(x + y) = \frac{f(x + y)}{f(0)} = \frac{f(x)f(y)(f(0))^{1986}}{f(0)} = \frac{f(x)}{f(0)} \cdot \frac{f(y)}{f(0)} \cdot (f(0))^{1987} = g(x)g(y).$$

It follows easily that $g(x) = e^{bx}$ for some complex number $b$. From this we get that $f(x) = f(0)e^{bx}$. Since the condition (ii) implies $e^{(b-\overline{b})(x-1988)} = 1$ we obtain that $b$ must be real. Answer: $f \equiv 0$ or $f(x) = ae^{bx}$ where $a^{1987} = 1$ and $b$ is real.

**3.** [1990: 5] *First Selection Test of the Chinese I.M.O. Team 1988.*

In triangle $ABC$, angle $C$ is $30°$. $D$ is a point on $AC$ and $E$ is a point on $BC$ such that $AD = BE = AB$. Prove that $OI = DE$ and $OI$ is perpendicular to $DE$, where $O$ and $I$ are respectively the circumcentre and incentre of triangle $ABC$.

*Solution by D.J. Smeenk, Zaltbommel, The Netherlands.*

We generalize the problem by considering an arbitrary triangle $ABC$ with $c < a$, $c < b$. The projection of $O$ on $CB$ is $F$, that on $CA$ is $G$. The projection of $I$ on $CB$ is $H$, that on $CA$ is $K$. Now

$$HF = BF - BH = \frac{1}{2}a - (s - b) = \frac{1}{2}(b - c) = \frac{1}{2}CD$$

and $KG = \frac{1}{2}(a-c) = \frac{1}{2}CE$. The production of $GO$ intersects $IH$ (or its production) in $L$. Consider triangle $OIL$. Angle $\angle ILO = \gamma$ $(= \angle C)$, since $OL \perp CA$, $IL \perp CB$. It is easy to verify that

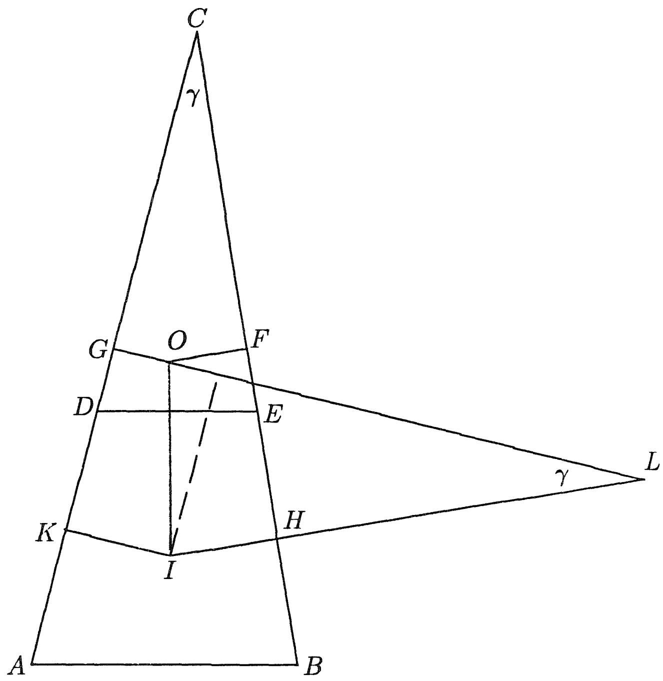
$$IL = \frac{KG}{\sin\gamma} = \frac{CE}{2\sin\gamma}$$

and

$$OL = \frac{HF}{\sin\gamma} = \frac{CD}{2\sin\gamma}.$$

We see that triangles $LIO$ and $CED$ are similar, for $LI : LO = CE : CD$. Moreover $LI \perp CE$ and $LO \perp CD$ imply $IO \perp ED$. Also

$$IO = \frac{DE}{2\sin\gamma},$$




and for $\gamma = 30°$ this gives $IO = DE$, as desired.

*

The last solution to a problem in the January number received from readers is the following.

**1.** [1990: 5] *Second Selection Test of the Chinese I.M.O. Team 1988.*

Define $x_n = 3x_{n-1} + 2$ for all positive integers $n$. Prove that an integer value can be chosen for $x_0$ such that 1988 divides $x_{100}$.

*Solutions by Seung-Jin Bang, Seoul, Republic of Korea; and by Murray S. Klamkin, University of Alberta.*

Letting $x_n = 3^n y_n$, we obtain the geometric difference equation $y_n = y_{n-1} + 2/3^n$. Thus

$$y_n = y_0 + \frac{2}{3}\left(1 + \frac{1}{3} + \cdots + \frac{1}{3^{n-1}}\right)$$

so that

$$x_n = (x_0 + 1)3^n - 1.$$

We now want $x_{100} = (x_0+1)3^{100} - 1$ to be divisible by 1988. Now let $3^{100} = 1988k + r$ where $0 < r < 1988$. Since 1988 is not divisible by 3, $r$ is relatively prime to 1988. It now suffices to show the existence of integers $x_0$ and $\ell$ such that $r(x_0+1) - 1 = 1988\ell$. Since $(r, 1988) = 1$ there are an infinite number of desired integer pairs $(x_0, \ell)$. Indeed since $1988 = 4 \cdot 7 \cdot 71$, we must have $x_{100} \equiv 0 \bmod 4$, $x_{100} \equiv 0 \bmod 7$ and $x_{100} \equiv 0 \bmod 71$; it follows that $x_0 \equiv 0 \bmod 4$, $x_0 \equiv 1 \bmod 7$, and $x_0 \equiv 45 \bmod 71$. By the Chinese Remainder Theorem we have $x_0 \equiv 400 \bmod 1988$.

* * *

We now turn to problems from the February 1990 number of the Corner. Here are solutions to the first five problems from the *XIV "ALL UNION" Mathematical Olympiad (U.S.S.R.)* [1990: 33–34]. A frequently occurring team solver is JACL, an acronym for two Edmonton students, Jason A. Colwell of Old Scona School and Calvin Li of Archbishop MacDonald School, and Andy Liu of The University of Alberta.

**1.** All the two-digit numbers from 19 to 80 are written in a row. The result is read as a single integer 19202122...787980. Is this integer divisible by 1980?

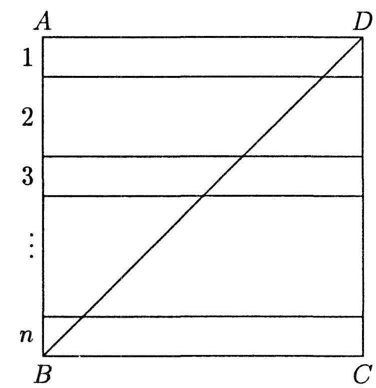
*Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by JACL; by Stewart Metchette, Culver City, California; by John Morvay, Springfield, Missouri; by Bob Prielipp, University of Wisconsin–Oshkosh; by Don St. Jean, George Brown College, Toronto; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Since $1980 = 2^2 \cdot 3^2 \cdot 5 \cdot 11$, we only need check divisibility by $4, 9, 5$ and $11$. Since the last two digits are 8 and 0, the number is divisible by both 4 and 5. The sum of the digits in the odd positions is $1 + (2+3+4+5+6+7)10 + 8 = 279$. The sum of those in the even positions is

$$9 + (0+1+2+3+4+5+6+7+8+9)6 + 0 = 279.$$

Since $279 + 279 = 558$ is divisible by 9, so is the number. Since $279 - 279 = 0$ is divisible by 11, so is the number. Hence the number is divisible by 1980.

**2.** Side $AB$ of a square $ABCD$ is divided into $n$ segments in such a way that the sum of lengths of the even numbered segments equals the sum of lengths of the odd numbered segments. Lines parallel to $AD$ are drawn through each point of division, and each of the $n$ "strips" thus formed is divided by diagonal $BD$ into a left region and a right region. Show that the sum of the areas of the left regions with odd numbers is equal to the sum of the areas of the right regions with even numbers.




*Solutions by JACL; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

We show that the conclusion holds when $ABCD$ is any rectangle. Let $AD = a$ and $AB = b$. Denote by $h_i$ the height of the $i$th strip, so that $\sum_{i \text{ odd}} h_i = \sum_{i \text{ even}} h_i$ and $\sum_{i=1}^{n} h_i = b$. Also let $L_i$ and $R_i$ denote the area of the $i$th left region and right region, respectively, $i = 1, 2, \ldots, n$. Then

$$\sum_{i \text{ odd}} L_i + \sum_{i \text{ even}} L_i = \frac{1}{2}ab$$

and

$$\sum_{i \text{ even}} L_i + \sum_{i \text{ even}} R_i = a \sum_{i \text{ even}} h_i = \frac{1}{2}ab.$$

From these we immediately get

$$\sum_{i \text{ odd}} L_i = \sum_{i \text{ even}} R_i.$$

**3.** A payload, packed into containers, is to be delivered to the orbiting space station "Salyut". There are at least 35 containers, and the total payload weighs exactly 18 tons. Seven "Progress" transport ships are available, each of which can deliver a 3-ton load. It is known that these ships altogether can (at least) carry any 35 of the containers at once. Show that in fact they can carry the entire load at once.

*Solution by JACL.*

Arrange the containers in non-increasing order of weight. Let $n$ be the largest integer such that all the containers from the first to the $n$th can be carried by the ships. By hypothesis $n \geq 35$. Suppose there is another container of weight $\omega$ tons. Since it cannot be carried as well, each ship must already be carrying more than $3-\omega$ tons. However, the total weight being carried so far is at most $18-\omega$. Hence $7(3-\omega) < 18-\omega$ or $\omega > 1/2$. Each of the $n$ containers being carried weighs at least $\omega$ tons, and $n \geq 35$. Counting the extra container, the total weight exceeds $36\omega > 18$ tons, which is a contradiction.

**4.** Points $M$ and $P$ are the midpoints of sides $BC$ and $CD$ of convex quadrilateral $ABCD$. If $AM + AP = a$, show that the area of the region $ABCD$ is less than $a^2/2$.

*Solution by JACL.*

Denote by $[P]$ the area of polygon $P$. Since $BM = CM$, $[ABM] = [ACM]$. Similarly $[ADP] = [ACP]$ so that $[ABCD] = 2[AMCP]$. Now $MP$ is parallel to $BD$, and $C$ is at the same distance from $MP$ as $BD$ is from $MP$. Since $ABCD$ is convex, $A$ is on the opposite side to $C$ of $BD$. Now $AMP$ and $CMP$ have the same base. Since $AMP$ has the greater altitude, $[AMP] > [CMP]$. Finally, note that

$$[AMP] = \frac{1}{2}AM \cdot AP \sin MAP \leq \frac{1}{2}AM \cdot AP \leq \frac{1}{8}a^2$$

by the Arithmetic-Mean Geometric-Mean Inequality. Hence

$$[ABCD] = 2[AMCP] = 2[AMP] + 2[CMP] < 4[AMP] \leq \frac{1}{2}a^2$$

as desired.

**5.** Does the equation $x^2 + y^3 = z^4$ have solutions for prime numbers $x, y$ and $z$?

*Solutions by Dieter Bennewitz, Koblenz, Germany; by JACL; by John Morvay, Springfield, Missouri; by Bob Prielipp, University of Wisconsin–Oshkosh; by Don St. Jean, George Brown College, Toronto; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. (The solution given is the one sent in by Bob Prielipp.)*

We shall show that the equation has *no* solutions for a positive integer $x$ and prime numbers $y$ and $z$.

Suppose on the contrary that there is a solution of this form. The equation is equivalent to

$$y^3 = (z^2 - x)(z^2 + x).$$

Because $y$ is a prime number and $x$ is a positive integer, $z^2 - x = 1$ and $z^2 + x = y^3$ or $z^2 - x = y$ and $z^2 + x = y^2$.

If $z^2 - x = 1$ and $z^2 + x = y^3$, then

$$2z^2 = y^3 + 1 = (y+1)(y^2 - y + 1).$$

It follows that $y \neq 2$, so that $y$ is an odd prime number. Thus $y > 2$, making $y+1 > 3$ and $y^2 - y + 1 > y + 1$. Since $z$ is a prime number we now obtain $z = y + 1$ and $y^2 - y + 1 = 2z$. But $z = y + 1$ where $y$ and $z$ are prime numbers with $y > 2$ is impossible.

If $z^2 - x = y$ and $z^2 + x = y^2$ then

$$2z^2 = y^2 + y = y(y+1).$$

It follows that $y \neq 2$, so $y$ is an odd prime. Thus $y > 2$. Since $y < y + 1$ and $z$ is a prime we get $y = z$ and $y + 1 = 2z$. This gives $z = 1$, a contradiction.

*Comment by Murray S. Klamkin, University of Alberta.*

There are however an infinite number of solutions if we remove the prime number restrictions, and even for the more general equation

$$x^r + y^s = z^t \quad \text{where} \quad (rs, t) = 1.$$

Just let $x = 2^{ms}$, $y = 2^{mr}$ and $z = 2^p$. Then we must satisfy $mrs + 1 = pt$. Since $rs$ is relatively prime to $t$, there exist an infinite number of positive integer pairs $(m, p)$ satisfying the conditions.

* * *

That's all the space available for this issue! Send me your contests and solutions.

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## BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

*Mathematical Challenges*, edited by the Scottish Mathematical Council, published by Blackie and Son Limited, Glasgow, 1989, ISBN 0-216-92622-X, softcover, 180 pages. *Reviewed by Andy Liu, University of Alberta.*

This book is a collection of the 213 problems set in the first twelve years of the "Mathematical Challenges", organized by the Scottish Mathematical Council. Unfortunately, the preface gives no description of the organization and philosophy of the "Mathematical Challenges". The readers should bear in mind that this is not a conventional contest, where participants write the papers in time-limited sittings. Instead, four sets of problems are sent out during the school year, and the students have over a month to think about them.

Many of the problems have a strong recreational flavour, a welcome relief from the usual "solid and stolid" mode of British offerings. There is a great variety, including cryptarithms, cross-numbers, logical inferences and number patterns. There is also a healthy dosage of problems in geometry, as well as in diophantine equations, inequalities and other Olympiad topics. Some of them are familiar classics, but most are new to the reviewer.

Here is a sample problem. "Four players successively select two cards from four numbered cards face down on a table. The total values of the cards drawn are 6, 9, 12 and 15. Then two of the four cards are turned over and their total value is found to be 11. Determine the value of each of the other two cards."

The problems are primarily at the pre-Olympiad level, even though a few prove rather difficult. The reviewer feels strongly that there is an urgent need for a book such as this. Many aspiring young students who may be scared off by the Olympiad level problems will find the ones in this book attractive. Once they have gained some proficiency and confidence, they will look for further challenges.

Although no names are associated with the book, it is no secret that the driving force behind the "Mathematical Challenges" is the indefatigable Dr. David Monk of the University of Edinburgh. To initiate such an enrichment program for the students is an admirable feat, but to have sustained the effort over such a long period displays conviction and dedication at the highest level.

For readers in North America, there is unfortunately no local distributor. All orders should be placed directly with Blackie and Son Limited at 7 Leicester Place, London, WC2H7BP, United Kingdom. The current price is £8.50 , plus £3.81 for air mail or £1.26 for surface mail.

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## PROBLEMS

*Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.*

*To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before* **June 1, 1992**, *although solutions received after that date will also be considered until the time when a solution is published.*

**1681**. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABC$ is an isosceles triangle with $\overline{AB} = \overline{AC} < \overline{BC}$. Let $P$ be a point on side $BC$ such that $\overline{AP}^2 = \overline{BC} \cdot \overline{PC}$, and let $CD$ be a diameter of the circumcircle of $\Delta ABC$. Prove that $\overline{DA} = \overline{DP}$.

**1682***. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For a finite set $S$ of natural numbers let

$$Alt(S) = x_1 - x_2 + x_3 - \cdots,$$

where $x_1 > x_2 > x_3 > \cdots$ are the elements of $S$ in decreasing order. Determine

$$f(n) = \sum Alt(S),$$

where the sum is extended over all non-empty subsets $S$ of $\{1, 2, \ldots, n\}$.

**1683**. *Proposed by P. Penning, Delft, The Netherlands.*

Given is a fixed triangle $ABC$ and fixed positive angles $\mu, \nu$ such that $\mu + \nu < \pi$. For a variable line $l$ through $C$, let $P$ and $Q$ be the feet of the perpendiculars from $A$ and $B$, respectively, to $l$, and let $Z$ be such that $\angle ZPQ = \mu$ and $\angle ZQP = \nu$ (and, say, the sense of $QPZ$ is clockwise). Determine the locus of $Z$.

**1684**. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Let

$$f(x, y, z) = x^4 + x^3 z + ax^2 z^2 + bx^2 y + cxyz + y^2.$$

Prove that for any real numbers $b, c$ with $|b| > 2$, there is a real number $a$ such that $f$ can be written as the product of two polynomials of degree 2 with real coefficients; furthermore, if $b$ and $c$ are rational, $a$ will also be rational.

**1685**. *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

If equilateral triangles $A_2A_3P_1, A_3A_1P_2, A_1A_2P_3$ are erected externally on the sides of a triangle $A_1A_2A_3$, then $A_1P_1, A_2P_2, A_3P_3$ concur at a point $R$ called the *isogonic center* (see p. 218 of R.A. Johnson, *Advanced Euclidean Geometry*). Prove that the line joining $R$ and its isogonal conjugate is parallel to the Euler line of the triangle.

**1686**. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

The sequence $a_0, a_1, a_2, \ldots$ is defined by $a_0 = 4/3$ and

$$a_{n+1} = \frac{3(5 - 7a_n)}{2(10a_n + 17)}$$

for $n \geq 0$. Find a formula for $a_n$ in terms of $n$.

**1687**. *Proposed by Jisho Kotani, Akita, Japan.*

The octahedron $ABCDEF$ is such that the three space diagonals $AF, BD, CE$ meet at right angles. Show that

$$[ABC]^2 + [ADE]^2 + [CDF]^2 + [BEF]^2 = [ACD]^2 + [ABE]^2 + [BCF]^2 + [DEF]^2,$$

where $[XYZ]$ is the area of triangle $XYZ$.

**1688**. *Proposed by Leroy F. Meyers, The Ohio State University.*

Solve the equation

$$a \log b = \log(ab)$$

if $a$ and $b$ are required to be (i) positive integers, (ii) positive rational numbers.

**1689**. *Proposed by Hidetosi Fukagawa, Aichi, Japan.*

$AA'$ is a diameter of circle $C = (O, r)$. Two congruent circles $C_1 = (O_1, a)$ and $C_2 = (O_2, a)$ $(a < r)$ are internally tangent to $C$ at $A$ and $A'$ respectively. In one half of the circle $C$ we draw two more circles $(O_3, b)$ and $(O_4, c)$ externally touching each other, both internally touching $C$, and also externally touching $C_1$ and $C_2$ respectively. Show that

$$\text{(i) } r = a + b + c\,; \qquad \text{(ii) } O_3O_4 \parallel AA'\,.$$

**1690**. *Proposed by Charlton Wang, student, Waterloo Collegiate Institute, and David Vaughan and Edward T.H. Wang, Wilfrid Laurier University.*

When working on a calculus problem, a student misinterprets "the average rate of change of $f(x)$ from $a$ to $b$" to mean "the average of the rates of change of $f(x)$ at $a$ and $b$", but obtains the correct answer. Determine all infinitely differentiable functions $f(x)$ for which this occurs, i.e., for which

$$\frac{f(b) - f(a)}{b - a} = \frac{f'(b) + f'(a)}{2}$$

for all $a \neq b$.

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## SOLUTIONS

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

**1564**. [1990: 205] *Proposed by Jordi Dou, Barcelona, Spain.*

Given three pairs of points $(P, P'), (Q, Q'), (R, R')$, each pair isogonally conjugate with respect to a fixed unknown triangle, construct the isogonal conjugate $X'$ of a given point $X$.

*Solution by the proposer.*

The construction relies on classical theorems that can be found in the older projective geometry texts or in Pierre Samuel's recent *Projective Geometry* (Springer, Undergraduate Texts in Mathematics, 1988), see especially sections 2.4 (pp. 64–65) and 4.2. The crux is that $X$ and $X'$ are isogonal conjugates with respect to $\Delta ABC$ if and only if they are conjugate with respect to each conic of the pencil $\Omega$ determined by the four points $I, I_A, I_B, I_C$, the incentre and excentres of $\Delta ABC$. These conics are equilateral hyperbolas. We know that the polars of a point $X$ with respect to the conics of a pencil all pass through a point $X'$; this is the conjugate of $X$ that we are after. In order to resolve the problem it is sufficient to determine two hyperbolas $\omega_1$, $\omega_2$ of $\Omega$ and construct the polars $x_1$, $x_2$ of $X$ with respect to $\omega_1$, $\omega_2$. Their intersection is then $X'$.

We put $PP' = p$, $QQ' = q$, $RR' = r$, $p \cap q = N$, $q \cap r = L$, $r \cap p = M$. Let $P_N$ and $P_M$ be the harmonics of $N$ and $M$ with respect to $P, P'$, and analogously define $Q_L$, $Q_N$ and $R_M$, $R_L$. Let $Q_N R_M = p'$, $R_L P_N = q'$, $P_M Q_L = r'$, and put $p' \cap q' = N'$, $q' \cap r' = L'$, $r' \cap p' = M'$. The pairs $PP'$, $QQ'$, $RR'$ are then conjugates with respect to every one of the degenerate conics $[p, p']$, $[q, q']$, $[r, r']$. All the conics which pass through $N$, $P_N$, $N'$, $Q_N$ (the intersections of $[p, p']$ and $[q, q']$) have as conjugate points the three pairs $PP', QQ', RR'$, and analogously for the conics which pass through $[q, q'] \cap [r, r'] = \{L, Q_L, L', R_L\}$ and through $[r, r'] \cap [p, p'] = \{M, R_M, M', P_M\}$. Let $H_1$ be the orthocentre of $NP_NQ_N$. The conic $\omega_1 = [NP_NN'Q_NH_1]$ is an equilateral hyperbola and has as conjugates the pairs $PP'$, $QQ'$, $RR'$, and analogously for the conic $\omega_2 = [LQ_LL'R_LH_2]$, $H_2$ being the orthocentre of $LQ_LR_L$.

Using Pascal's theorem, construct the other intersections $Y_1$ and $Z_1$ of the lines $XN'$ and $XP_N$ with $\omega_1$. The polar $x_1$ of $X$ with respect to $\omega_1$ is the line joining the points $Y_1Z_1 \cap N'P_N$ and $Y_1P_N \cap Z_1N'$. Analogously construct the polar $x_2$ of $X$ with respect to $\omega_2$. Finally, $X' = x_1 \cap x_2$.

Since all constructions in the solution are with straightedge alone, there exists a unique solution in general. On the other hand, the determination of the quadrilateral $II_AI_BI_C = \omega_1 \cap \omega_2$ (and of the diagonal points $A, B, C$) involves finding the points of intersection of two conics — i.e., solving a fourth degree equation. They cannot generally be constructed with Euclidean tools.

I have not found a solution for the analogous problem where "isogonal" is replaced by "isotomic".

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**1567.** [1990: 266] *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Let
$$f(x_1, x_2, \ldots, x_n) = \frac{x_1\sqrt{x_1 + \cdots + x_n}}{(x_1 + \cdots + x_{n-1})^2 + x_n} .$$
Prove that $f(x_1, x_2, \ldots, x_n) \le \sqrt{2}$ under the condition that $x_1 + \cdots + x_n \ge 2$ and all $x_i \ge 0$.

*Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $k \ge 1$. We consider more generally $f(x_1, \ldots, x_n)$ where $x_i \ge 0$ for all $i$ and $x_1 + \cdots + x_n \ge k$, and show
$$f(x_1, \ldots, x_n) \le \frac{1}{2 - 1/\sqrt{k}} .$$
Put $a = x_1$, $b = x_2 + \cdots + x_{n-1}$, $c = x_n$. Then
$$f(x_1, \ldots, x_n) = \frac{a\sqrt{a+b+c}}{(a+b)^2 + c} =: g(a, b, c),$$
where $a, b, c \ge 0$, $a + b + c \ge k$. We put $a + b + c = s$, $s \ge k$, $s$ fixed for the moment. Then
$$g(a, b, c) = \frac{a\sqrt{s}}{(s-c)^2 + c} \le \frac{(s-c)\sqrt{s}}{(s-c)^2 + c} = \frac{\sqrt{s}}{(s-c) + c/(s-c)} .$$

Now
$$(s-c)+\frac{c}{s-c} \geq 2\sqrt{s}-1,$$
as this inequality is equivalent to $(c-s+\sqrt{s})^2 \geq 0$. Therefore
$$g(a,b,c) \leq \frac{\sqrt{s}}{2\sqrt{s}-1} = \frac{1}{2-1/\sqrt{s}}\,.$$
Since the term on the right hand side decreases, we get
$$g(a,b,c) \leq \frac{1}{2-1/\sqrt{k}}\,,$$
as was to be shown. There holds equality for $b=0$, $c=k-\sqrt{k}$, $a=\sqrt{k}$, i.e. for $x_1=\sqrt{k}$, $x_2=\cdots=x_{n-1}=0$, $x_n=k-\sqrt{k}$.

For the original problem ($k=2$) we get the better bound
$$f(x_1,\ldots,x_n) \leq \frac{\sqrt{2}}{2\sqrt{2}-1} = \frac{4+\sqrt{2}}{7}\,.$$

*Also solved by MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; JOHN H. LINDSEY, Northern Illinois University, DeKalb; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

*Kuczma, Lau and Lindsey all obtained the best possible bound given above.*

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**1568.** [1990: 205] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Show that
$$\sum \sin A \geq \frac{2}{\sqrt{3}}\left(\sum \cos A\right)^2$$
where the sums are cyclic over the angles $A, B, C$ of an acute triangle.

I. *Solution by Vedula N. Murty, Penn State Harrisburg.*

The inequality proposed can be restated as
$$y \geq \frac{2}{\sqrt{3}}\,(1+x)^2$$
where
$$y=\frac{s}{R}=\sum \sin A, \quad 1+x=1+\frac{r}{R}=\sum \cos A,$$
$s, r, R$ being the semiperimeter, inradius and circumradius respectively. In the paper "A new inequality for $R$, $r$, and $s$", published in *Crux* [1982: 62–68], I have presented a picture showing the Type I and Type II triangle regions. The curve
$$y=\frac{2}{\sqrt{3}}\,(1+x)^2 \tag{1}$$

is a parabola opening upwards with vertex at the point $M(-1,0)$ shown in my picture. It cuts the $y$-axis at $(0, 2/\sqrt{3})$, which is below the point $E(0,\sqrt{3})$, and passes through $A(1/2, 3\sqrt{3}/2)$. Thus the inequality clearly holds for all Type I triangles since the corresponding region (vertical shading) is above the curve (1). It is also clear that there exist obtuse Type II triangles for which the proposed inequality does not hold. However, from the classic Steinig inequality

$$y^2 \geq 16x - 5x^2$$

(item 5.8 in Bottema et al, *Geometric Inequalities*), we see that the proposed inequality holds for all $x$ values for which

$$16x - 5x^2 \geq \frac{4}{3}(1+x)^4,$$

i.e.,

$$4x^4 + 16x^3 + 39x^2 - 32x + 4 \leq 0.$$

The above polynomial has two real roots $x_0 \approx 0.157$ and $x_1 = 0.5$, and two complex conjugate roots. Hence for $x_0 \leq x \leq 1/2$ the proposed inequality holds. [*Editor's note.* From Murty's paper, all acute Type II triangles correspond to values $x \geq 1/4$, thus all acute Type II triangles satisfy the proposed inequality. Alternatively, since point $F$ in Murty's picture has $y$–coordinate $\sqrt{71 - 8\sqrt{2}}/4$, one need only check that

$$\frac{2}{\sqrt{3}}(1+\frac{1}{4})^2 < \frac{1}{4}\sqrt{71+8\sqrt{2}},$$

since then the curve (1) must pass below $F$.]

II. *Solution by the proposer.*

From

$$\sum \cos A = 1 + 4\prod \sin(A/2)$$

we have

$$\begin{aligned}(\sum \cos A)^2 &= 1 + 8\prod \sin(A/2) + 16\prod \sin^2(A/2)\\ &\leq 1 + 8\prod \sin(A/2) + 16(1/8)\prod \sin(A/2)\\ &= 1 + 10\prod \sin(A/2).\end{aligned}$$

So a sharper inequality to prove is

$$\sum \sin A \geq \frac{2}{\sqrt{3}}\left(1 + 10\prod \sin(A/2)\right). \tag{2}$$

Now

$$\sum \sin A = \frac{s}{R} \quad \text{and} \quad \prod \sin(A/2) = \frac{r}{4R}$$

so from (2) we have to prove that $\sqrt{3}s \geq 2R + 5r$, or

$$3s^2 \geq 4R^2 + 20Rr + 25r^2.$$

It is known (*Crux* 999 [1986: 80]) that for acute triangles,

$$s^2 \geq 2R^2 + 8Rr + 3r^2,$$

so it suffices to show that

$$6R^2 + 24Rr + 9r^2 \geq 4R^2 + 20Rr + 25r^2,$$

which simplifies to

$$(R+4r)(R-2r) = R^2 + 2Rr - 8r^2 \geq 0,$$

which is true.

*Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; and JOHN H. LINDSEY, Northern Illinois University, DeKalb.*

*Kuczma also showed that the inequality holds for some obtuse triangles, namely for any triangle whose angles do not exceed* $\approx 132^\circ\ 40'$. *Janous made the substitution* $A \longrightarrow (\pi - A)/2$, *etc., in the given inequality to obtain*

$$\sum \cos\frac{A}{2} \geq \frac{2}{\sqrt{3}}\left(\sum \sin\frac{A}{2}\right)^2 ,$$

*valid for all triangles.*

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**1569**. [1990: 205] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Evaluate

$$\int_0^a \frac{\ln(1+ax)}{1+x^2}dx$$

where $a$ is a constant.

I. *Solution by Margaret Izienicki, Mount Royal College, Calgary.*

Let $F(a)$ be the given integral. By differentiation with respect to $a$ under the integral sign, we get

$$\begin{aligned}
\frac{d}{da}F(a) &= \int_0^a \frac{x\,dx}{(1+ax)(1+x^2)} \ + \ \frac{\ln(1+a^2)}{1+a^2} \qquad (F(0)=0)\\
&= \frac{1}{1+a^2}\int_0^a \left(\frac{x}{1+x^2} + \frac{a}{1+x^2} - \frac{a}{1+ax}\right)dx \ + \ \frac{\ln(1+a^2)}{1+a^2}\\
&= \frac{1}{1+a^2}\left(\frac{1}{2}\ln(1+a^2) + a\tan^{-1}a\right)\\
&= \frac{d}{da}\left(\frac{1}{2}\ln(1+a^2)\cdot\tan^{-1}a\right).
\end{aligned}$$

Integrating with respect to $a$, and applying $F(0) = 0$, we get

$$F(a) = \frac{1}{2}\ln(1+a^2)\cdot\tan^{-1}a.$$

II. *Solution by Maria Mercedes Sánchez Benito, I.B. Luis Bunuel, Alcorcón, Madrid, Spain.*

If we made the change

$$x = \frac{a-u}{1+au},$$

then

$$dx = \frac{-(1+au)-(a-u)a}{(1+au)^2}\,du = -\frac{1+a^2}{(1+au)^2}\,du,$$
$$1+ax = 1+\frac{a(a-u)}{1+au} = \frac{1+a^2}{1+au},$$

and

$$1+x^2 = 1+\left(\frac{a-u}{1+au}\right)^2 = \frac{1+2au+a^2u^2+a^2-2au+u^2}{(1+au)^2} = \frac{(1+a^2)(1+u^2)}{(1+au)^2}.$$

Also,

$$x=0 \implies u=a,$$
$$x=a \implies a(1+au)=a-u \implies u=0.$$

Thus

$$\begin{aligned}\int_0^a \frac{\log(1+ax)}{1+x^2}\,dx &= \int_a^0 \frac{\log[(1+a^2)/(1+au)]}{1+u^2}(-du)\\ &= \int_0^a \frac{\log(1+a^2)}{1+u^2}\,du - \int_0^a \frac{\log(1+au)}{1+u^2}\,du\\ &= \log(1+a^2)\arctan a - \int_0^a \frac{\log(1+au)}{1+u^2}\,du,\end{aligned}$$

so

$$\int_0^a \frac{\log(1+ax)}{1+x^2}\,dx = \frac{1}{2}\log(1+a^2)\arctan a.$$

*Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ILIA BLASKOV, Technical University, Gabrovo, Bulgaria; LEN BOS, University of Calgary; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa,* *Poland; KEE-WAI LAU, Hong Kong; JOHN H. LINDSEY, Northern Illinois University, DeKalb; BEATRIZ MARGOLIS, Paris, France; JEAN-MARIE MONIER, Lyon, France; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. There was one partial solution received.*

*The proposer remembered hearing the integral from his calculus professor in Taiwan in 1961. Hess found the integral in Gradshteyn and Ryzhik, p. 556, no. 4.29.18. Janous located it in the Russian book Prudnikov, Brychkov, Marichev,* Integrals and Series (Elementary Functions), *p. 506, item 2.6.11.1.*

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**1570.** [1990: 205] *Proposed by P. Penning, Delft, The Netherlands.*

In $n$-dimensional space it is possible to arrange $n+1$ $n$-dimensional solid spheres of unit radius in such a way that they all touch one another. Determine the radius of the small solid sphere that touches all $n+1$ of these spheres.

*Solution by John H. Lindsey, Northern Illinois University, DeKalb.*

The $(n+1)$–tuples

$$(1,0,\dots,0),(0,1,0,\dots,0),\ \dots\ ,(0,\dots,0,1)$$

are mutually distance $\sqrt{2}$ apart and really lie in $\mathbf{R}^n$ since they lie in the hyperplane $x_1+x_2+\cdots+x_{n+1}=1$. The center of mass

$$\left(\frac{1}{n+1},\dots,\frac{1}{n+1}\right)$$

is distance

$$\sqrt{\left(1-\frac{1}{n+1}\right)^2+n\left(\frac{1}{n+1}\right)^2}=\sqrt{\frac{n}{n+1}}$$

from the original points. Blowing up the scale by a factor $\sqrt{2}$ to make the original $n+1$ points mutually distance 2 apart, this becomes $\sqrt{2n/(n+1)}$ so the center ball should have radius

$$\sqrt{\frac{2n}{n+1}}-1.$$

[Lindsey ended by noting that the result is known and referring to a book of C.A. Rogers, probably *Packing and covering*, Cambridge Univ. Press, 1964, p. 79. —*Ed.*]

*Also solved by EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; CARLES ROMERO CHESA, I.B. Manuel Blancafort, La Garriga, Catalonia, Spain; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.*

*    *    *    *    *

**1571**. [1990: 239] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $ABC$ be a triangle with circumradius $R$ and area $F$, and let $P$ be a point in the same plane. Put $AP = R_1$, $BP = R_2$, $CP = R_3$, $R'$ the circumradius of the pedal triangle of $P$, and $p$ the power of $P$ relative to the circumcircle of $\Delta ABC$. Prove that

$$18R^2R' \geq a^2R_1 + b^2R_2 + c^2R_3 \geq 4F\sqrt{3|p|}.$$

*Combination of solutions of Murray S. Klamkin, University of Alberta, and the proposer.*

By using the following known properties [1] of the pedal triangle,

$$\text{sides } (a', b', c') = \left(\frac{aR_1}{2R}, \frac{bR_2}{2R}, \frac{cR_3}{2R}\right),$$

$$\text{area } F' = \frac{|R^2 - \overline{OP}^2|F}{4R^2} = \frac{|p|F}{4R^2},$$

the proposed inequalities can be written in the more revealing form

$$9RR' \geq aa' + bb' + cc' \geq 4\sqrt{3FF'}\,. \tag{1}$$

(It is to be noted that the formula for $F'$ is derived in [1] assuming $P$ is an interior point. However, it can be shown that if $P$ is an exterior point then this formula remains valid.)

We now show that (1) is valid for any two arbitrary triangles of sides $(a, b, c)$ and $(a', b', c')$. The right hand inequality is just *Crux* 1114 [1987: 185]. For the left hand inequality, we use the known inequality $9R^2 \geq a^2 + b^2 + c^2$ (see [2]) to give, by Cauchy's inequality,

$$3R \cdot 3R' \geq \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{a'^2 + b'^2 + c'^2} \geq aa' + bb' + cc',$$

with equality if and only if both triangles are equilateral.

*References:*

[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, N.Y., 1960, pp. 136, 139.

[2] O. Bottema et al, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969, item 5.13.

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**1572**. [1990: 239] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Consider a "signed harmonic series"

$$\sum_{n=1}^{\infty} \frac{\epsilon_n}{n}, \quad \epsilon_n = \pm 1 \text{ for each } n.$$

Assuming that plus and minus signs occur with equal frequency, i.e.

$$\lim_{n\to\infty} \frac{\epsilon_1 + \cdots + \epsilon_n}{n} = 0,$$

prove or disprove that the series necessarily converges.

I. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The answer is negative. Indeed, let $\mathbf{P}$ be the set of primes. Then consider

$$\mathbf{N}\backslash\mathbf{P} = \{1, 4, 6, 8, 9, 10, 12, \ldots\} =: \{n_1, n_2, n_3, \ldots\}.$$

As an alternating series, $\sum_{k=1}^{\infty}(-1)^k/n_k$ converges, whereas $\sum_{p\in\mathbf{P}} 1/p$ diverges. So, putting $\epsilon_{n_k} = (-1)^k$ for all $k = 1, 2, \ldots$ and $\epsilon_p = 1$ for all $p \in \mathbf{P}$,

$$\sum_{n=1}^{\infty} \frac{\epsilon_n}{n} = \sum_{p\in\mathbf{P}} \frac{1}{p} + \sum_{k=1}^{\infty} \frac{(-1)^k}{n_k}$$

diverges. But

$$\frac{\pi(n) - 1}{n} \le \frac{\epsilon_1 + \cdots + \epsilon_n}{n} \le \frac{\pi(n)}{n},$$

where $\pi(n)$ is the number of primes $\le n$. Since $\pi(n) \sim n/\log n$, we infer

$$\lim_{n\to\infty} \frac{\epsilon_1 + \cdots + \epsilon_n}{n} = 0.$$

[*Editor's note*: the above proof refers to the Prime Number Theorem ($\pi(n) \sim n/\log n$), but only needs the consequence $\pi(n)/n \longrightarrow 0$.]

II. *Generalization by Robert B. Israel, University of British Columbia.*

It does not necessarily converge. More generally, the following is true.

THEOREM. Consider any sequence $a_n \ge 0$ such that $\sum_{n=1}^{\infty} a_n = \infty$. Then there is a sequence $(\epsilon_n)$ such that $\epsilon_n = \pm 1$,

$$\lim_{n\to\infty} \frac{\epsilon_1 + \cdots + \epsilon_n}{n} = 0, \quad \text{and} \quad \sum_{n=1}^{\infty} \epsilon_n a_n \quad \text{diverges.}$$

*Proof.* Take a sequence of integers $(t_k)$ such that $t_0 = 0$, and for all $k$,

$$\sum_{n=t_k+1}^{t_{k+1}} a_n \ge 2k + 1,$$
$$t_{k+1} - t_k \ge 9(t_k - t_{k-1}),$$

and $t_{k+1} - t_k$ is a positive multiple of $2k + 1$. Let

$$S_k = \sum_{n=t_k+1}^{t_{k+1}} a_n \quad (\ge 2k + 1).$$

We have $S_k = \sum_{j=0}^{2k} T_{jk}$ where

$$T_{jk} = \sum\{a_n : t_k + 1 \le n \le t_{k+1},\ n \equiv j \bmod 2k + 1\}.$$

[$T_{jk}$ is the sum of every $(2k+1)$st term of $S_k$, starting at the $j$th.] Let $U_k$ be the subset of $\{0,1,\ldots,2k\}$ corresponding to the $k+1$ largest $T_{jk}$ values (breaking ties arbitrarily). Thus

$$\sum_{j\in U_k} T_{jk} \geq \frac{k+1}{2k+1} S_k.$$

If $t_k+1 \leq n \leq t_{k+1}$, let $\epsilon_n = +1$ if $n \pmod{2k+1} \in U_k$, and $-1$ otherwise. We have

$$\sum_{n=t_k+1}^{t_{k+1}} \epsilon_n a_n = \sum_{j\in U_k} T_{jk} - \sum_{j\notin U_k} T_{jk} \geq \frac{k+1}{2k+1} S_k - \frac{k}{2k+1} S_k = \frac{S_k}{2k+1} \geq 1,$$

so that the series $\sum \epsilon_n a_n$ diverges. Next we will estimate $\sum_{n=1}^{N} \epsilon_n$. Note that in any $2k+1$ consecutive $n$'s from $t_k+1$ to $t_{k+1}$ there will be $k+1$ positive and $k$ negative $\epsilon_n$, adding to 1. Thus

$$\sum_{n=t_k+1}^{t_{k+1}} \epsilon_n = \frac{t_{k+1}-t_k}{2k+1}.$$

On the other hand, if $t_k+1 \leq N \leq t_{k+1}$, we have $N = t_k + q(2k+1) + r$ for some $q$ and $r$ with $0 \leq r < 2k+1$. Then

$$\begin{aligned}\left|\sum_{n=1}^{N} \epsilon_n\right| &\leq \sum_{j=0}^{k-1}\left|\sum_{n=t_j+1}^{t_j+1} \epsilon_n\right| + \sum_{i=0}^{q-1}\left|\sum_{n=t_k+i(2k+1)+1}^{t_k+(i+1)(2k+1)} \epsilon_n\right| + \left|\sum_{n=t_k+q(2k+1)+1}^{t_k+q(2k+1)+r} \epsilon_n\right| \\ &\leq \sum_{j=0}^{k-1} \frac{t_{j+1}-t_j}{2j+1} + q + r.\end{aligned}$$

Note that for $j \leq k-1$,

$$\begin{aligned}\frac{t_{j+1}-t_j}{2j+1} &\leq \frac{9^{j-k+1}}{2j+1}(t_k - t_{k-1}) = \frac{3^j}{2j+1} 3^{j-2k+2}(t_k - t_{k-1}) \\ &\leq \frac{3^{k-1}}{2k-1} 3^{j-2k+2}(t_k - t_{k-1}) = \frac{3^{j-k+1}}{2k-1}(t_k - t_{k-1}),\end{aligned}$$

since $3j/(2j+1)$ is an increasing function of $j$. This implies

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} \epsilon_n\right| &\leq \frac{3(t_k - t_{k-1})}{(2k-1)N} \sum_{j=0}^{k-1}\left(\frac{1}{3}\right)^{k-j} + \frac{q}{N} + \frac{r}{N} \\ &< \frac{3}{2(2k-1)} \frac{t_k - t_{k-1}}{N} + \frac{q}{q(2k+1)} + \frac{2k+1}{t_k - t_{k-1}} \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty,\end{aligned}$$

since $N > t_k \geq t_k - t_{k-1} \geq 9^{k-1}$ for all $k$.

*Also solved by MURRAY S. KLAMKIN, University of Alberta; JIAQI LUO, Cornell University, Ithaca, N.Y.; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. There was one incorrect solution submitted.*

*The proposer's solution was the same as Janous's. The other three solutions used the fact that the series $\sum(n\log n)^{-1}$ diverges.*

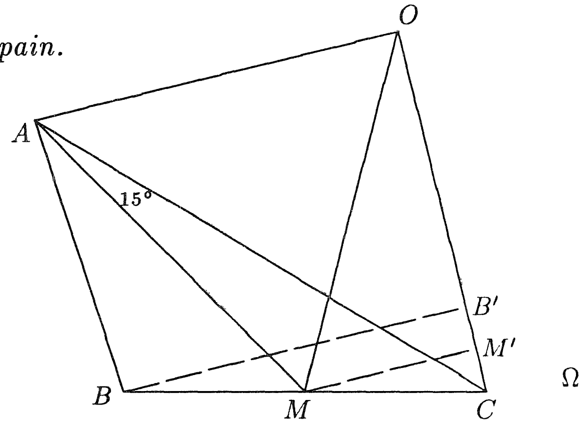
* * * * *

**1573**. [1990: 239] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let $M$ be the midpoint of $BC$ of a triangle $ABC$. Suppose that $\angle BAM = \angle C$ and $\angle MAC = 15°$. Calculate angle $C$.

I. *Solution by Jordi Dou, Barcelona, Spain.*

Let $\Omega$ be the circle through $AMC$, with centre $O$ and radius $R$. Since $\angle MAC = 15°$, $\angle MOC = 30°$. Since $\angle MCA = \angle MAB$, $AB$ will be tangent to $\Omega$ at $A$. The distance $MM'$ of $M$ to $AC$ is $R\sin 30° = R/2$, and since $CB = 2CM$ the distance $BB'$ of $B$ to $AC$ will be $R = AO$. Thus $AB$ is parallel to $OC$, so $\angle AOC = 90°$ and $\angle C = \angle BAM = \angle BAC - \angle MAC = 45° - 15° = 30°$.




II. *Solution by Dag Jonsson, Uppsala, Sweden.*

Let $\alpha = \angle C = \angle BAM$, $c = |AB|$ and $a = |BM| = |MC|$. Since $\Delta ABC$ is similar to $\Delta MBA$, $c/(2a) = a/c$, i.e. $c = \sqrt{2}a$. The sinus theorem applied to $\Delta ABC$ gives

$$\frac{2a}{\sin(\alpha + 15°)} = \frac{\sqrt{2}a}{\sin\alpha},$$

so

$$\frac{\sin(\alpha+15°)}{\sin\alpha} = \frac{1/\sqrt{2}}{1/2} = \frac{\sin(30°+15°)}{\sin 30°},$$

with the obvious solution $\alpha = 30°$. It is unique since

$$\frac{\sin(\alpha+15°)}{\sin\alpha} = \cos 15° + \frac{\sin 15°}{\tan\alpha}$$

is decreasing in $\alpha$ (here $\alpha < 90°$).

*Also solved by HAYO AHLBURG, Benidorm, Spain; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; MARGARET IZIENICKI, Mount Royal College, Calgary; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; MARIA MERCEDES SÁNCHEZ BENITO, I.B. Luis Bunuel, Madrid, Spain; K.R.S. SASTRY, Addis Ababa, Ethiopia;*

*D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY (two solutions), Williamsburg, Virginia; L.J. UPTON, Mississauga, Ontario; ALBERT W. WALKER, Toronto, Ontario; KENNETH M. WILKE, Topeka, Kansas; and the proposer. Two incorrect solutions were also sent in.*

*Janous and Walker gave generalizations.*

* * * * *

**1574**. [1990: 239] *Proposed by Murray S. Klamkin, University of Alberta.*

Determine sharp upper and lower bounds for the sum of the squares of the sides of a quadrilateral with given diagonals $e$ and $f$. For the upper bound, it is assumed that the quadrilateral is convex.

*Solution by Hayo Ahlburg, Benidorm, Spain.*

The required upper and lower bounds are respectively

$$2(e^2+ef+f^2) \quad \text{and} \quad e^2+f^2.$$

In the quadrilateral $ABCD$ where $U$, $V$ are the midpoints of the diagonals $AC = e$ and $BD = f$ and the point of intersection of the diagonals is $I$, we have the relationship (attributed by Carnot to Euler)

$$(AB)^2+(BC)^2+(CD)^2+(DA)^2 = e^2+f^2+4(UV)^2$$

(see N. Altshiller Court, *College Geometry*, pp. 126 and 300). The lower bound $e^2+f^2$ is reached for $UV = 0$, $U \equiv V \equiv I$. $ABCD$ is then a parallelogram. The upper bound is approached, not reached, if $UV$ grows as far as possible for given values of $e$ and $f$ Looking at $\Delta UVI$, we see that

$$UV < UI + IV < UA + DV = \frac{e+f}{2} \,,$$

so that

$$(AB)^2+(BC)^2+(CD)^2+(DA)^2 < 2(e^2+ef+f^2).$$

As $A$ and $D$ get closer and $\angle UIV$ approaches 180°, we get as close as we want to this upper bound. It would only be reached by a degenerate "quadrilateral", a straight line $BVAIDUC$, with $A \equiv I \equiv D$.

*Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; and the proposer. One incorrect solution was received.*

*The proposer notes that the lower bound provides an immediate proof of a conjecture of Tutescu, referred to on p. 411 of D.S. Mitrinović et al,* Recent Advances in Geometric Inequalities.

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**1575***. [1990: 240] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The rational number 5/2 has the property that when written in decimal expansion, i.e. 2,5 (2.5 in North America), there appear exactly the (base 10) digits of numerator and denominator in permuted form. Do there exist infinitely many $m, n \in \mathbf{N}$, neither ending in 0, so that $m/n$ has the same property?

I. *Solution by Robert B. Israel, University of British Columbia.*

Yes. The easiest infinite family of solutions is

$$\frac{59}{2} = 29.5, \ \frac{599}{2} = 299.5, \ \frac{5999}{2} = 2999.5, \ \cdots ,$$

i.e., $5(9)_k/2 = 2(9)_k.5$, $k = 1, 2, 3, \cdots$ . Here $(b)_k$ denotes a block of digits $(b)$ repeated $k$ times. For convenience I'm using $ab$ to denote the concatenation of blocks of digits $a$ and $b$; I'll use $\times$ for multiplication. There are also infinitely many solutions with $n = 5$, e.g.,

$$\frac{6028124(9)_k}{5} = 1205624(9)_k.8 \ .$$

The real question, I think, is whether there are solutions for infinitely many different $n$. I conjecture that *for every odd positive integer $t$ there are infinitely many solutions for $n = 2^t$*. (It is not hard to show that there are no solutions for $n = 5^t$ with $t > 1$, because $m/n$ will have fewer digits than required.)

Some examples with $n = 2^t$, $t$ odd:

$$t = 3: \quad \frac{22514(285714)_k}{8} = 2814(285714)_k.25$$

$$t = 5: \quad \frac{7052}{32} = 220.375$$

$$t = 7: \quad \frac{20516}{128} = 160.28125$$

$$t = 9: \quad \frac{6983312}{512} = 13639.28125$$

$$t = 11: \quad \frac{3781162256}{2048} = 1846270.6328125$$

Two partial results:

LEMMA 1. There are no solutions for $n = 2^t$ where $t$ is even.

*Proof.* Write the equation as $m/n = x/10^\ell$ where $x$ is an integer containing precisely the digits of $m$ and $n$. Since each integer is congruent mod 9 to the sum of its digits, we would have $m + n \equiv x$ as well as $m \equiv n \times x$, so that

$$(n-1) \times x = n \times x - x \equiv m - (m+n) = -n \bmod 9.$$

But if $n = 2^t$ with $t$ even, $n - 1$ is divisible by 3 while $n$ is not, so this is impossible. $\square$

LEMMA 2. If there is at least one solution $(m, n)$ for a given $n = 2^t$ with $m \geq n+8$, then there are infinitely many.

*Proof.* Consider a solution

$$\frac{Ab}{n} = \frac{X}{10^\ell},$$

where $A$ is an integer and $b$ a single nonzero digit, and $X$ is an integer containing precisely all digits of $A$, $n$ and $b$ (we could get along without the $b$ except for the rather artificial restriction that $m$ should not end in 0). Let $A = n \times Y + r$ where $0 \leq r < n$. Then

$$\begin{aligned} X &= 10^\ell \times \frac{10 \times A + b}{n} = 10^{\ell+1} \times Y + 10^\ell \times \frac{10 \times r + b}{n} \\ &= YZ \quad \text{ where } Z = 10^\ell \times \frac{10 \times r + b}{n} < 10^{\ell+1} \text{ has } \ell + 1 \text{ digits.} \end{aligned}$$

Then for a nonnegative integer $C$ of $d$ digits (with leading zeros allowed), we will obtain a new solution

$$\frac{ACb}{n} = \frac{YCZ}{10^\ell},$$

i.e.

$$\frac{10^{d+1} \times A + 10 \times C + b}{n} = 10^{d+1} \times Y + 10 \times C + Z \times 10^{-\ell},$$

if

$$(10^d - 1) \times r = (n-1) \times C. \tag{1}$$

Take $d > 0$ so that $10^d \equiv 1 \bmod (n-1)$ ($d$ exists because $(10, 2^t - 1) = 1$ when $t$ is odd), and let

$$C = r \times \frac{10^d - 1}{n - 1};$$

since $0 \leq r < n$, $C < 10^d$, so this $C$ will have at most $d$ digits, and (1) is satisfied.

The condition $m \geq n+8$ ensures that $YCZ$ has no leading zeros. For, if $Ab = m < 10 \times n$ we get $Y = 0$ and $r = A$ and the new solution would be

$$\frac{ACb}{n} = \frac{CZ}{10^\ell}.$$

But then (since $m = Ab = rb = 10 \times r + b$)

$$r = \frac{m-b}{10} \geq \frac{n-1}{10}$$

so that

$$C = \frac{r \times (10^d - 1)}{n-1} \geq \frac{10^d - 1}{10}.$$

Since $C$ is an integer, this means that $C \geq 10^{d-1}$ and there are no leading zeros in $C$. $\square$

[*Editor's note.* The condition "neither $m$ nor $n$ ends in 0" was meant by the editor to prevent such trivial solutions as $50/2 = 25.0, 500/2 = 25.00, \cdots$ and $5/20 = 0.25, 5/200 = 0.025, \cdots$. There was probably a better way to do this.]

II. *Solution by Richard K. Guy, University of Calgary.*

Rather trivially: $5/2, 59/2, 599/2, 5999/2, \cdots$ .

If recurring decimals are allowed:

$$\frac{8572}{3} = 2857.\overline{3}, \quad \frac{8571428572}{3} = 2857142857.\overline{3},$$

$$\frac{8571428571428572}{3} = 2857142857142857.\overline{3}, \quad \cdots ,$$

also

$$\frac{816639344262295081967}{123} = 6639344262295081967.\overline{21138},$$

$$\frac{816639344262295081967213}{123} = 6639344262295081967213.\overline{11382}$$

(what is the next member of this sequence?),

$$\frac{3007}{429} = 7.\overline{009324}, \quad \frac{30070}{429} = 70.\overline{093240}, \quad \frac{300700}{429} = 700.\overline{932400}$$

(what is the next member of this sequence?),

$$\frac{56168}{819} = 68.\overline{581196}$$

(does this belong to an infinite family of solutions?), and so on.

*Except for one reader who misinterpreted the problem, no other solutions were sent in. Readers can make amends by settling Israel's conjecture or by finding other "repeating decimal" solutions à la Guy!*

*A related problem, proposed by Michael Runge (of Winnipeg) recently appeared in the* Journal of Recreational Mathematics. *See number 1862, p. 68 of the 1991 volume.*

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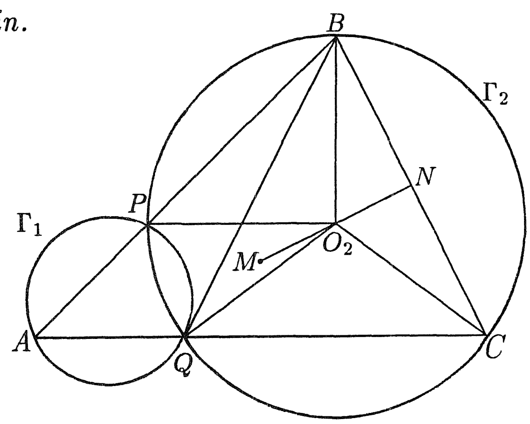
**1576.** [1990: 240] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Circles $\Gamma_1$ and $\Gamma_2$ have a common chord $PQ$. $A$ is a variable point of $\Gamma_1$. $AP$ and $AQ$ intersect $\Gamma_2$ for the second time in $B$ and $C$ respectively. Show that the circumcentre of $\Delta ABC$ lies on a fixed circle. (This problem is not new. A reference will be given when the solution is published.)

I. *Solution by José Yusty Pita, Madrid, Spain.*




Let $N$ be the midpoint of $BC$ and $M$ the circumcenter of $\Delta ABC$. $M$ belongs to the perpendicular bisector of $BC$, so

$$MN = NB \cdot \cot A.$$

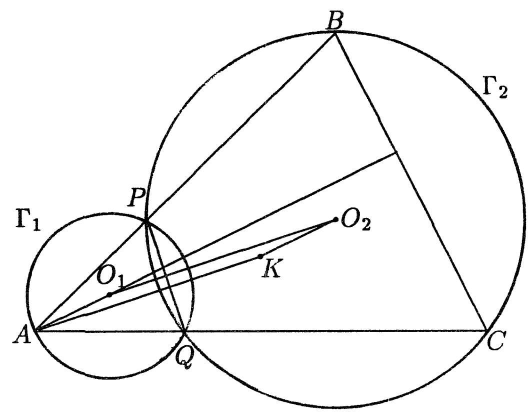
But $NB = BC/2$ is constant because, with $O_2$ the center of $\Gamma_2$,

$$\angle A = \angle BQC - \angle PBQ = \frac{\angle BO_2C - \angle PO_2Q}{2},$$

and $\angle A$ and $\angle PO_2Q$ are constant. Therefore $MN$ is constant, and $O_2N$ is also constant (since $BC$ is). So $MO_2 = MN - O_2N$ is constant and the locus of $M$ is a circle concentric with $O_2$.

II. *Solution by C. Festraets-Hamoir, Brussels, Belgium.*

Soit $K$ le centre du cercle circonscrit à $ABC$ et soient $O_1, O_2$, les centres respectifs de $\Gamma_1$ et $\Gamma_2$. L'inversion de pôle $A$ et de puissance $AP \cdot AB = AQ \cdot AC$ (i) transforme le cercle $\Gamma_1$ en la droite $BC$, d'où $AO_1 \perp BC$, et (ii) transforme le cercle circonscrit au triangle $ABC$ en la droite $PQ$, d'où $AK \perp PQ$. On a ainsi $AO_1 \parallel KO_2$ et $AK \parallel O_1O_2$, donc $AO_1O_2K$ est un parallélogramme et $\overrightarrow{AK} = \overrightarrow{O_1O_2}$, quelle que soit la position de $A$ sur $\Gamma_1$. Le lieu de $K$ est l'image du cercle $\Gamma_1$ par la translation $\overrightarrow{O_1O_2}$.




*Also solved by JORDI DOU, Barcelona, Spain (two solutions); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.*

*Most solvers observed that the circle on which the circumcentre of $\Delta ABC$ lies is congruent to $\Gamma_1$. This nice result is in Solution* II, *but it seems (?) it can't be deduced from Solution* I. *Of Dou's two solutions, his first was like Solution* I, *and his second gave the same conclusion as Solution* II, *with some extra properties of the diagram as well.*

*The proposer found the problem (without solution) in Aref and Wernick,* Problems and Solutions in Euclidean Geometry, *Dover, New York, Exercise* 6.17.

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**1577***. [1990: 240] *Proposed by Isao Ashiba, Tokyo, Japan.*

Suppose $\alpha, \beta, \gamma$ are arbitrary angles such that $\cos\alpha \neq \cos\beta$, and $x$ is a real number such that

$$x^2 \cos\beta\cos\gamma + x(\sin\beta + \sin\gamma) + 1 = 0$$

and

$$x^2 \cos\gamma\cos\alpha + x(\sin\gamma + \sin\alpha) + 1 = 0.$$

Prove that

$$x^2 \cos\alpha\cos\beta + x(\sin\alpha + \sin\beta) + 1 = 0.$$

*Solution by P. Penning, Delft, The Netherlands.*

Note that $C$ has to be eliminated from the first two equations. This is simple with the aid of the "trick" of solving first for $\sin C$ and $\cos C$. By subtracting, the equations give

$$x(\cos A - \cos B)\cos C = -(\sin A - \sin B); \tag{1}$$

by multiplying the first equation by $\cos A$ and the second by $\cos B$ and subtracting, they give

$$x(\cos A - \cos B)\sin C = -(\cos A - \cos B) + x\sin(A - B). \tag{2}$$

Note that both results become identities for $A = B$. Introduce

$$s = \frac{A+B}{2}, \quad v = \frac{A-B}{2},$$

and use

$$-2\sin s \sin v = \cos A - \cos B, \quad 2\cos s \sin v = \sin A - \sin B;$$

then (1) and (2) become, after division by $\sin v$,

$$\begin{aligned} x\sin s\cos C &= \cos s, \\ x\sin s\sin C &= -\sin s - x\cos v. \end{aligned}$$

Elimination of $C$ is now simple by squaring and adding:

$$\begin{aligned} x^2\sin^2 s &= x^2\sin^2 s(\cos^2 C + \sin^2 C) \\ &= \cos^2 s + \sin^2 s + 2x\sin s\cos v + x^2\cos^2 v \\ &= 1 + 2x\sin s\cos v + x^2\cos^2 v, \end{aligned}$$

or

$$x^2(\cos^2 v - \sin^2 s) + 2x\sin s\cos v + 1 = 0,$$

or

$$x^2\cos(v+s)\cos(v-s) + x(\sin(s+v) + \sin(s-v)) + 1 = 0.$$

Introducing $A$ and $B$ again yields the desired result.

*Also solved by HAYO AHLBURG, Benidorm, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; G.P. HENDERSON, Campbellcroft, Ontario; JOHN H. LINDSEY, Ft. Myers, Florida; and JEAN-MARIE MONIER, Lyon, France.*

*Bellot notes that the problem appears (without solution) in Hobson's* Plane Trigonometry, *Fifth Edition, Cambridge Univ. Press, 1921, Chapter 6, Example 11, p. 95.*

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**1578**. [1990: 240] *Proposed by O. Johnson and C.S. Goodlad, students, King Edward's School, Birmingham, England.*

For each fixed positive real number $a_n$, maximise

$$\frac{a_1 a_2 \ldots a_{n-1}}{(1+a_1)(a_1+a_2)(a_2+a_3)\ldots(a_{n-1}+a_n)}$$

over all positive real numbers $a_1, a_2, \ldots, a_{n-1}$.

*Solution by Chris Wildhagen, Rotterdam, The Netherlands.*
Let
$$P = \frac{a_1 a_2 \dots a_{n-1}}{(1+a_1)(a_1+a_2)\dots(a_{n-1}+a_n)} .$$
$P$ can be rewritten as $P = Q^{-1}$, where

$$Q = (1+b_1)(1+b_2)\dots(1+b_n),$$

with $b_1 = a_1$ and $b_i = a_i/a_{i-1}$ for $2 \le i \le n$. Note that $b_1 b_2 \dots b_n = a_n$. We have to minimize $Q$. To achieve this, we use the following [Hölder's] inequality:

$$\sqrt[n]{(u_1+v_1)(u_2+v_2)\dots(u_n+v_n)} \ge \sqrt[n]{u_1 u_2 \dots u_n} + \sqrt[n]{v_1 v_2 \dots v_n} ,$$

where $u_i, v_i > 0$ for all $i$ and equality holds if and only if

$$\frac{u_1}{v_1} = \frac{u_2}{v_2} = \dots = \frac{u_n}{v_n} .$$

Applying this result gives

$$\sqrt[n]{Q} \ge 1 + \sqrt[n]{b_1 b_2 \dots b_n} = 1 + a_n^{1/n},$$

hence
$$Q \ge (1 + a_n^{1/n})^n,$$
with equality if and only if $b_1 = b_2 = \dots = b_n = a_n^{1/n}$, i.e.,

$$a_i = a_n^{i/n}, \quad 1 \le i \le n-1. \tag{1}$$

We can conclude that
$$P \le (1 + a_n^{1/n})^{-n},$$
with equality if and only if (1) holds.

*Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; and the proposers. Two other readers sent in upper bounds for the given expression which were not attained for all values of $a_n$.*

*Klamkin gave a generalization.*

* * * * *

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